Further studies on relic neutrino asymmetry generation II: a rigorous treatment of repopulation in the adiabatic limit

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Abstract

We derive an approximate relic neutrino asymmetry evolution equation that systematically incorporates repopulation processes from the full quantum kinetic equations (QKEs). It is shown that in the collision dominant epoch, the said equation reduces precisely to the expression obtained previously from the static/adiabatic approximation. The present treatment thus provides a rigorous justification for the seemingly incongruous assumptions of a negligible repopulation function and instantaneous repopulation sometimes employed in earlier works.

I. INTRODUCTION

The computationally convenient and rewarding static approximation [1] may be rigorously derived from the quantum kinetic equations (QKEs) [2,3] in the adiabatic Boltzmann limit [4,5], assuming that the repopulation or refilling function is insignificantly small. In actual numerical studies, however, we also adopt the view that a depleted neutrino momentum state is instantaneously refilled from the background plasma. The latter condition naturally implies that while the repopulation function may be small, it cannot be identically zero.

In principle, simultaneous suppositions of a vanishing refilling function and instantaneous repopulation are somewhat contradictory, or at least uncontrolled. However, their combined validity in the collision dominant epoch is strongly supported by numerical evidence [1,5]. Indeed, instantaneous repopulation *without* the assumption of a negligible repopulation function can be consistently implemented as a sensible approximation to the QKEs [6].

Commencing with the full-fledged QKEs, we demonstrate rigorously that the two aforementioned approximations, conceived originally for computational convenience, are indeed

appropriate in a certain limit. Our approach differs from the fully consistent instantaneous repopulation approximation introduced in Ref. [6], since we do not make any *a priori* assumptions regarding the repopulation rate, instantaneous or otherwise. Instead, the present work shows that a "brute force" treatment of the QKEs incorporating a finite repopulation rate does indeed lead to approximate evolution equations that reduce to those following from the dual approximations of a negligible refilling function and instantaneous repopulation in the temperature regime of interest.

II. NOMENCLATURE

Consider a two-flavour system comprising an active species ν_α (where $\alpha = e$, μ or τ), and a sterile species ν_s, whose properties are characterised by the density matrix [2,3]

$$\rho(p) = \frac{1}{2}[P_0(p) + \mathbf{P}(p) \cdot \sigma], \tag{1}$$

in which the variables P_0 and $\mathbf{P}(p) = P_x(p)\hat{x} + P_y(p)\hat{y} + P_z(p)\hat{z}$ are functions of both time t and momentum p, and $\sigma = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$ are the Pauli matrices. The diagonal entries of ρ represent respectively the ν_α and ν_s distribution functions in momentum space, that is,

$$N_\alpha(p) = \frac{1}{2}[P_0(p) + P_z(p)]N^{\text{eq}}(p, 0),$$
$$N_s(p) = \frac{1}{2}[P_0(p) - P_z(p)]N^{\text{eq}}(p, 0), \tag{2}$$

where the reference distribution $N^{\text{eq}}(p, 0)$ is defined as the equilibrium Fermi–Dirac function,

$$N^{\text{eq}}(p, \mu) = \frac{1}{2\pi^2} \frac{p^2}{1 + \exp\left(\frac{p-\mu}{T}\right)}, \tag{3}$$

with chemical potential $\mu = 0$ at temperature T.

The evolution of $P_0(p)$ and $\mathbf{P}(p)$ is governed by the quantum kinetic equations (QKEs)

$$\frac{\partial \mathbf{P}}{\partial t} = \mathbf{V}(p) \times \mathbf{P}(p) - D(p)[P_x(p)\hat{x} + P_y(p)\hat{y}] + \frac{\partial P_0}{\partial t}\hat{z},$$
$$\frac{\partial P_0}{\partial t} = R_\alpha(p). \tag{4}$$

Here, the quantity $\mathbf{V}(p) = \beta(p)\hat{x} + \lambda(p)\hat{z}$ is the matter potential vector [7], with

$$\beta(p) = \frac{\Delta m^2}{2p} \sin 2\theta_0,$$
$$\lambda(p) = \frac{\Delta m^2}{2p}[b(p) - a(p) - \cos 2\theta_0], \tag{5}$$

in which Δm^2 is the mass-squared difference between the neutrino states, θ_0 is the vacuum mixing angle, and

$$a(p) = -\frac{4\zeta(3)\sqrt{2}G_F L^{(\alpha)}T^3 p}{\pi^2 \Delta m^2},$$

$$b(p) = -\frac{4\zeta(3)\sqrt{2}G_F A_\alpha T^4 p^2}{\pi^2 \Delta m^2 m_W^2}, \tag{6}$$

given that G_F is the Fermi constant, m_W the W-boson mass, ζ the Riemann zeta function and $A_e \simeq 17$, $A_{\mu,\tau} \simeq 4.9$. The effective total lepton number (for the α-neutrino species)

$$L^{(\alpha)} = L_\alpha + L_e + L_\mu + L_\tau + \eta \equiv 2L_\alpha + \tilde{L}, \tag{7}$$

combines all asymmetries individually defined as $L_\alpha = (n_{\nu_\alpha} - n_{\overline{\nu}_\alpha})/n_\gamma$, where the symbol n denotes number density, and η is a small term due to the cosmological baryon asymmetry.

The decoherence function $D(p)$ is related to the total collision rate for ν_α, $\Gamma(p)$, via [2]

$$D(p) = \frac{1}{2}\Gamma(p) = \frac{1}{2}\frac{p}{\langle p \rangle_0} y_\alpha G_F^2 T^5, \tag{8}$$

where $\langle p \rangle_0 \simeq 3.15T$ is the average momentum for a relativistic Fermi–Dirac distribution with zero chemical potential, $y_e \simeq 4$, $y_{\mu,\tau} \simeq 2.9$ and $z_e \simeq 0.1$, $z_{\mu,\tau} \simeq 0.04$.

The repopulation or refilling function

$$R_\alpha(p) \simeq \Gamma(p)\left\{K_\alpha - \frac{1}{2}[P_0(p) + P_z(p)]\right\}, \tag{9}$$

with

$$K_\alpha \equiv \frac{N^{\rm eq}(p,\mu)}{N^{\rm eq}(p,0)}, \tag{10}$$

is determined from assuming thermal equilibrium for all species in the background plasma, and that the ν_α distribution is also approximately thermal [4]. Physically, Eq. (9) means that as active neutrinos of some momentum are converted to sterile neutrinos, a gap is created in the Fermi–Dirac distribution. The weak interaction processes repopulate this depleted state, driving the ensemble back towards equilibrium.

A separate but equivalent set of expressions parameterises the evolution of the $\overline{\nu}_\alpha \leftrightarrow \overline{\nu}_s$ system. These are distinguished from their ordinary counterparts with an overhead bar.

III. STANDARD STATIC/ADIABATIC LIMIT: A BRIEF OUTLINE

Together with the requirement of $\alpha + s$ lepton number conservation, we may derive from the QKEs an exact evolution equation for the neutrino asymmetry L_α, that is [4],

$$\frac{dL_\alpha}{dt} = \frac{1}{2n_\gamma} \int \beta[P_y(p) - \overline{P}_y(p)]N^{\rm eq}(p,0)dp, \tag{11}$$

where the quantities P_y and $\overline{P}_y$ are obtained from numerically integrating the respective QKEs for the neutrino and antineutrino systems given by Eq. (4). The role of the

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static/adiabatic limit approximation [1,4,5], therefore, is to generate approximate expressions for P_y that are dynamically decoupled from the other variables in order to minimise the computational effort.

The first step in the formal adiabatic procedure of Ref. [4] consists of setting the repopulation function to zero, i.e.,

$$R_\alpha \simeq 0, \tag{12}$$

thereby reducing the four-component QKEs to a system of three coupled homogeneous differential equations. Further manipulation produces the approximate equality

$$P_y(t) \simeq -\frac{\beta D}{D^2 + \lambda^2} P_z(t), \tag{13}$$

in the limit $D, |\lambda| \gg |\beta|$, such that Eq. (11) becomes[1]

$$\frac{dL_\alpha}{dt} \simeq \frac{1}{2n_\gamma} \int \beta^2 \left[\frac{\overline{D}(\overline{N}_\alpha - \overline{N}_s)}{\overline{D}^2 + \overline{\lambda}^2} - \frac{D(N_\alpha - N_s)}{D^2 + \lambda^2} \right] dp. \tag{14}$$

Detailed discussions of the standard adiabatic procedure may be found in the companion paper Ref. [5].

In numerical studies, an analogous expression describing sterile neutrino production for each momentum state is employed for tracking the quantity $N_s(p)$ in Eq. (14), while the ν_α distribution function is taken to be

$$N_\alpha(p) \simeq N^{\mathrm{eq}}(p, \mu). \tag{15}$$

This is the so-called instantaneous repopulation approximation, which assumes that a depleted momentum state is immediately refilled from the background medium so that thermal equilibrium is always maintained.

A naïve interpretation of Eqs. (9) and (15) suggests that instantaneous repopulation leads directly to an identically zero refilling rate, which seems to argue for consistency between the central assumptions contained in Eqs. (12) and (15). However, the instantaneous repopulation limit is also related to taking the collision rate to infinity. Thus the right hand side of Eq. (9) is an *a priori* undetermined finite and generally nonzero function [6], that is apparently at odds with the assumption of a vanishing repopulation rate.

IV. EXTENDED ADIABATIC APPROXIMATION

The exact QKEs in Eq. (4) are fundamentally a system of four coupled first order differential equations, of both the homogeneous and inhomogeneous varieties, that is best displayed in matrix form,

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[1]The denominator in Eq. (13), and therefore Eq. (14), should properly contain an additive term β^2 to ensure the equations' validity even when the condition $D \gg |\beta|$ does not hold (see companion paper Ref. [5]). In this work, however, we shall assume that the said condition is always met, and omit the β^2 term for simplicity.

$$\frac{\partial}{\partial t}\begin{pmatrix} P_x \\ P_y \\ P_z \\ P_0 \end{pmatrix} = \begin{pmatrix} -D & -\lambda & 0 & 0 \\ \lambda & -D & -\beta & 0 \\ 0 & \beta & -D & -D \\ 0 & 0 & -D & -D \end{pmatrix}\begin{pmatrix} P_x \\ P_y \\ P_z \\ P_0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2DK_\alpha \\ 2DK_\alpha \end{pmatrix} \equiv \mathcal{K}^{\mathrm{rp}}\mathbf{P}^{\mathrm{rp}} + \mathbf{A}. \tag{16}$$

Dealing firstly with the homogeneous part of Eq. (16), we introduce an instantaneous diagonal basis onto which we map the vector $\mathbf{P}^{\mathrm{rp}} \equiv (\mathbf{P}, P_0)$ from its original fixed basis via the transformation

$$\begin{pmatrix} Q_1^{\mathrm{rp}} \\ Q_2^{\mathrm{rp}} \\ Q_3^{\mathrm{rp}} \\ Q_4^{\mathrm{rp}} \end{pmatrix} \equiv \mathbf{Q}^{\mathrm{rp}} = \mathcal{U}_{\mathrm{rp}}\mathbf{P}^{\mathrm{rp}}, \tag{17}$$

where, by definition,

$$\mathcal{K}_d^{\mathrm{rp}} \equiv \mathrm{diag}\,(\Lambda_1,\ \Lambda_2,\ \Lambda_3,\ \Lambda_4) = \mathcal{U}_{\mathrm{rp}}\mathcal{K}^{\mathrm{rp}}\mathcal{U}_{\mathrm{rp}}^{-1}, \tag{18}$$

such that the matrix $\mathcal{K}_d^{\mathrm{rp}}$ is diagonal and similar to $\mathcal{K}^{\mathrm{rp}}$.

The eigenvalues Λ_i are solutions to the quartic characteristic equation

$$\Lambda^4 + 4D\Lambda^3 + (5D^2 + \lambda^2 + \beta^2)\Lambda^2 + 2D(D^2 + \lambda^2 + \beta^2)\Lambda + \beta^2 D^2 = 0, \tag{19}$$

formally given by

$$\Lambda_i = -D \pm \frac{\sqrt{D^2 - \lambda^2 - \beta^2 \pm \sqrt{D^4 + 2D^2\lambda^2 + \lambda^4 + 2\beta^2\lambda^2 - 2\beta^2 D^2 + \beta^4}}}{\sqrt{2}}. \tag{20}$$

We are primarily interested in the limit $D, |\lambda| \gg |\beta|$, in which case the eigenvalues are well approximated by the leading order terms in the small β power series expansion of Eq. (20),

$$\Lambda_1 = \Lambda_2^* = -D + i\lambda + \frac{i}{2}\frac{\beta^2\lambda}{D^2 + \lambda^2} + \mathcal{O}(\beta^4),$$

$$\Lambda_3 = -\frac{1}{2}\frac{\beta^2 D}{D^2 + \lambda^2} + \mathcal{O}(\beta^4),$$

$$\Lambda_4 = -2D + \frac{1}{2}\frac{\beta^2 D}{D^2 + \lambda^2} + \mathcal{O}(\beta^4). \tag{21}$$

The transformation matrix $\mathcal{U}_{\mathrm{rp}}^{-1}$ comprises the column eigenvectors κ_i^{rp}, which for $\lambda \neq 0$ are exactly

$$\kappa_i^{\mathrm{rp}} = \begin{pmatrix} 1 \\ -\frac{D+\Lambda_i}{\lambda} \\ -\frac{\beta(D+\Lambda_i)^2}{\lambda\Lambda_i(2D+\Lambda_i)} \\ \frac{\beta D(D+\Lambda_i)}{\lambda\Lambda_i(2D+\Lambda_i)} \end{pmatrix}, \tag{22}$$

while its inverse $\mathcal{U}_{\mathrm{rp}}$ consists of the row vectors

$$v_i^{\mathrm{rp}} = \Lambda_i(2D + \Lambda_i) \left(\frac{1}{D^2} \prod_{j \neq i} \frac{D + \Lambda_j}{\Lambda_i - \Lambda_j}, \quad -\lambda \prod_{j \neq i} \frac{1}{\Lambda_i - \Lambda_j}, \right.$$

$$\frac{\lambda}{\beta D^2} \prod_{j \neq i} \frac{1}{\Lambda_i - \Lambda_j} \left[4D^3 + 2D^2(\Lambda_l + \Lambda_m + \Lambda_n) + D(\Lambda_l \Lambda_m + \Lambda_m \Lambda_n + \Lambda_n \Lambda_l) + \Lambda_l \Lambda_m \Lambda_n \right],$$

$$\left. \frac{\lambda}{\beta D} \prod_{j \neq i} \frac{1}{\Lambda_i - \Lambda_j} \left[4D^2 + 2D(\Lambda_l + \Lambda_m + \Lambda_n) + \Lambda_l \Lambda_m + \Lambda_m \Lambda_n + \Lambda_n \Lambda_l \right] \right), \tag{23}$$

where the indices l, m and n are the three integers not equal to i, e.g., for $i = 1$, l, m and n are 2, 3 and 4 respectively.

Note that Eqs. (22) and (23) do not apply if two or more eigenvalues are degenerate, in which case the matrix $\mathcal{K}^{\mathrm{rp}}$ may genuinely have less than four distinct eigenvectors thereby rendering $\mathcal{U}_{\mathrm{rp}}^{-1}$ momentarily uninvertible, or, if four linearly independent eigenvectors exist, alternative methods are needed for their evaluation. However, such circumstances are hard to come by and have practically no influence on the outcome of this work. Thus the exact QKEs [Eq. (16)] are equivalently

$$\frac{\partial \mathbf{Q}^{\mathrm{rp}}}{\partial t} = \mathcal{K}_d^{\mathrm{rp}} \mathbf{Q}^{\mathrm{rp}} + \mathcal{U}_{\mathrm{rp}} \mathbf{A} - \mathcal{U}_{\mathrm{rp}} \frac{\partial \mathcal{U}_{\mathrm{rp}}^{-1}}{\partial t} \mathbf{Q}^{\mathrm{rp}}, \tag{24}$$

written in the new instantaneous basis where the transformation matrices are valid.

The term $\mathcal{U}_{\mathrm{rp}} \frac{\partial \mathcal{U}_{\mathrm{rp}}^{-1}}{\partial t}$ in Eq. (24) contains explicit dependence on the derivatives of the parameters D, λ and β. These are discarded in the adiabatic limit (see Ref. [4] for the relevant constraints[2]) so that the remaining terms form a set of four decoupled inhomogeneous differential equations given by

$$\frac{\partial \mathbf{Q}^{\mathrm{rp}}}{\partial t} \simeq \mathcal{K}_d^{\mathrm{rp}} \mathbf{Q}^{\mathrm{rp}} + \mathbf{B}, \tag{25}$$

where $\mathbf{B} \equiv \mathcal{U}_{\mathrm{rp}} \mathbf{A}$, or equivalently in index form,

$$\frac{\partial}{\partial t} Q_i^{\mathrm{rp}}(t) \simeq \Lambda_i(t) Q_i^{\mathrm{rp}}(t) + B_i(t), \quad i = 1, 2, \ldots, 4. \tag{26}$$

Equation (26) may be formally solved to give

$$Q_i^{\mathrm{rp}}(t) = e^{\int_0^t \Lambda_i(t')dt'} Q_i^{\mathrm{rp}}(0) + e^{\int^t \Lambda_i(t')dt'} \mathcal{I}_i, \tag{27}$$

with the inhomogeneous segment of the QKEs contained entirely in the integral

$$\mathcal{I}_i = \int_0^t e^{-\int^{t'} \Lambda_i(t'')dt''} B_i(t')dt', \tag{28}$$

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[2]The bounds in Ref. [4] are calculated assuming $\frac{\partial P_0}{\partial t} \simeq 0$, and should naturally be different from those one would obtain with the incorporation of a finite repopulation function. However, we expect the difference to be minimal given the similarity between the relevant entries in the two cases' respective transformation matrices in the appropriate limit.

thus amounting to the following solution for the vector $\mathbf{P}^{\mathrm{rp}}$,

$$
\begin{pmatrix} P_x(t) \\ P_y(t) \\ P_z(t) \\ P_0(t) \end{pmatrix} = \mathcal{U}_{\mathrm{rp}}^{-1}(t) \left[\begin{pmatrix} e^{\int_0^t \Lambda_1(t')dt'} & 0 & 0 & 0 \\ 0 & e^{\int_0^t \Lambda_2(t')dt'} & 0 & 0 \\ 0 & 0 & e^{\int_0^t \Lambda_3(t')dt'} & 0 \\ 0 & 0 & 0 & e^{\int_0^t \Lambda_4(t')dt'} \end{pmatrix} \begin{pmatrix} Q_1^{\mathrm{rp}}(0) \\ Q_2^{\mathrm{rp}}(0) \\ Q_3^{\mathrm{rp}}(0) \\ Q_4^{\mathrm{rp}}(0) \end{pmatrix} \right.
$$
$$
\left. + \begin{pmatrix} e^{\int^t \Lambda_1(t')dt'} & 0 & 0 & 0 \\ 0 & e^{\int^t \Lambda_2(t')dt'} & 0 & 0 \\ 0 & 0 & e^{\int^t \Lambda_3(t')dt'} & 0 \\ 0 & 0 & 0 & e^{\int^t \Lambda_4(t')dt'} \end{pmatrix} \begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \\ \mathcal{I}_4 \end{pmatrix} \right], \tag{29}
$$

in the adiabatic limit.

Observe in Eq. (21) that the real components of the eigenvalues Λ_1, Λ_2 and Λ_4 are of the order D, while Λ_3 is proportional to β^2. This implies that the exponentials $\exp[\int_0^t \Lambda_i(t')dt']$, where $i = 1, 2, 4$, are rapidly damped relative to the "decay" time scale of their Λ_3 counterpart in the homogeneous part of Eq. (29) (for a full discussion, see Ref. [5]). We may therefore implement in Eq. (29) the collision dominance approximation

$$
e^{\int_0^t \Lambda_i(t')dt'} \to 0, \quad i = 1, 2, 4, \tag{30}
$$

to obtain

$$
P_y(t) \simeq -\frac{D + \Lambda_3}{\lambda} e^{\int_0^t \Lambda_3(t')dt'} Q_3^{\mathrm{rp}}(0) - \sum_{i=1}^4 \frac{D + \Lambda_i}{\lambda} e^{\int^t \Lambda_i(t')dt'} \mathcal{I}_i,
$$
$$
P_z(t) \simeq -\frac{\beta(D + \Lambda_3)^2}{\lambda \Lambda_3(2D + \Lambda_3)} e^{\int_0^t \Lambda_3(t')dt'} Q_3^{\mathrm{rp}}(0) - \sum_{i=1}^4 \frac{\beta(D + \Lambda_i)^2}{\lambda \Lambda_i(2D + \Lambda_i)} e^{\int^t \Lambda_i(t')dt'} \mathcal{I}_i,
$$
$$
P_0(t) \simeq \frac{\beta D(D + \Lambda_3)}{\lambda \Lambda_3(2D + \Lambda_3)} e^{\int_0^t \Lambda_3(t')dt'} Q_3^{\mathrm{rp}}(0) + \sum_{i=1}^4 \frac{\beta D(D + \Lambda_i)}{\lambda \Lambda_i(2D + \Lambda_i)} e^{\int^t \Lambda_i(t')dt'} \mathcal{I}_i, \tag{31}
$$

and consequently, from combining the above expressions for P_y and P_z,

$$
P_y(t) = \frac{\Lambda_3(2D + \Lambda_3)}{\beta(D + \Lambda_3)} P_z(t)
$$
$$
+ \sum_{j=1,2,4} \left(\frac{\Lambda_3(2D + \Lambda_3)}{\beta(D + \Lambda_3)} - \frac{\Lambda_j(2D + \Lambda_j)}{\beta(D + \Lambda_j)} \right) \frac{\beta(D + \Lambda_j)^2}{\lambda \Lambda_j(\Lambda_j + 2D)} e^{\int^t \Lambda_j(t')dt'} \mathcal{I}_j. \tag{32}
$$

Note that the collision dominance approximation in Eq. (30) is not immediately applicable to the indefinite integrals associated with the inhomogeneous part of Eq. (29) since the term $\mathcal{I}_i$ contains the factor $\exp[-\int^t \Lambda_i(t')dt']$.

With the quantity B_i in Eq. (28) evaluating to

$$
B_i = 2K_\alpha \frac{\lambda \Lambda_i(2D + \Lambda_i)}{\beta D} \prod_{j \neq i} \frac{2D + \Lambda_j}{\Lambda_i - \Lambda_j}, \tag{33}
$$

courtesy of Eq. (23), Eq. (32) simplifies to

$$P_y(t) = -\frac{\beta D}{D^2 + \lambda^2}P_z(t) - e^{-\int^t(D-i\lambda)dt'}\int_0^t e^{\int^{t'}(D-i\lambda)dt''}\frac{\beta DK_\alpha}{D + i\lambda}dt'$$

$$-e^{-\int^t(D+i\lambda)dt'}\int_0^t e^{\int^{t'}(D+i\lambda)dt''}\frac{\beta DK_\alpha}{D - i\lambda}dt'$$

$$+\frac{4D}{\lambda}e^{-\int^t 2Ddt'}\int_0^t e^{\int^{t'}2Ddt''}\frac{\beta\lambda DK_\alpha}{D^2 + \lambda^2}dt' + \mathcal{O}(\beta^3), \tag{34}$$

as a power series in the small expansion parameter β. Clearly, the first term in the above expression is simply the standard adiabatic result given by Eq. (13).

The second item in Eq. (34) involving the inhomogeneous parameter K_α may be partially solved using integration by parts,

$$e^{-\int^t(D-i\lambda)dt'}\int_0^t e^{\int^{t'}(D-i\lambda)dt''}\frac{\beta DK_\alpha}{D + i\lambda}dt'$$

$$= e^{-\int^t(D-i\lambda)dt'}\int_0^t (D - i\lambda)e^{\int^{t'}(D-i\lambda)dt''}\frac{\beta DK_\alpha}{D^2 + \lambda^2}dt'$$

$$= \frac{\beta DK_\alpha}{D^2 + \lambda^2}\bigg|_{t'=t} - e^{-\int_0^t(D-i\lambda)dt'}\frac{\beta DK_\alpha}{D^2 + \lambda^2}\bigg|_{t'=0}$$

$$-e^{-\int^t(D-i\lambda)dt'}\int_0^t e^{\int^{t'}(D-i\lambda)dt''}\left[\frac{\beta D}{D^2 + \lambda^2}\frac{dK_\alpha}{dt'} + K_\alpha\frac{d}{dt'}\left(\frac{\beta D}{D^2 + \lambda^2}\right)\right]dt'. \tag{35}$$

A sufficiently large damping parameter D again allows the approximation of $\exp[-\int_0^t(D - i\lambda)dt'] \to 0$, thereby promptly obliterating the second term on the right hand side of the last equality. The quantity $\frac{d}{dt}\left(\frac{\beta D}{D^2+\lambda^2}\right)$ in the integral turns out to be one of the elements constituting the 4×4 matrix $\mathcal{U}_{\text{rp}}\frac{\partial \mathcal{U}_{\text{rp}}}{\partial t}^{-1}$. Consistency with the adiabatic approximation then calls for the setting of $\frac{d}{dt}\left(\frac{\beta D}{D^2+\lambda^2}\right) \simeq 0$ such that solutions to $\mathbf{P}_{\text{rp}}$ have no explicit dependence on the time derivatives of the parameters D, λ and β.

Further exploitation of the powerful technique of integration by parts on Eq. (35) in conjunction with the aforementioned approximations thus yields

$$e^{-\int^t(D-i\lambda)dt'}\int_0^t e^{\int^{t'}(D-i\lambda)dt''}\frac{\beta DK_\alpha}{D + i\lambda}dt'$$

$$\simeq \frac{\beta DK_\alpha}{D^2 + \lambda^2}\bigg|_{t'=t} + \left[\sum_{n=1}^N\left(\frac{-1}{D - i\lambda}\right)^n\frac{\beta D}{D^2 + \lambda^2}\frac{d^{(n)}}{dt'^{(n)}}K_\alpha\right]_{t'=t}$$

$$-e^{-\int^t(D-i\lambda)dt'}\int_0^t e^{\int^{t'}(D-i\lambda)dt''}\left(\frac{-1}{D - i\lambda}\right)^N\frac{\beta D}{D^2 + \lambda^2}\frac{d^{(N+1)}}{dt'^{(N+1)}}K_\alpha dt'. \tag{36}$$

Above the neutrino decoupling temperature, the function K_α is well approximated by

$$K_\alpha \simeq 1 + \frac{12\zeta(3)}{\pi^2}\frac{e^{p/T}}{1 + e^{p/T}}L_\alpha, \tag{37}$$

for small L_α, and consequently,

$$\frac{d^{(n)}}{dt^{(n)}}K_\alpha \simeq \frac{12\zeta(3)}{\pi^2}\frac{e^{p/T}}{1+e^{p/T}}\frac{d^{(n)}}{dt^{(n)}}L_\alpha, \tag{38}$$

with the recognition that the dimensionless factor $e^{p/T}/(1+e^{p/T})$ is independent of time. Since the quantity $\frac{dL_\alpha}{dt}$ is of the order of β^2, the first term in the sum containing this is equivalently an $\mathcal{O}(\beta^3)$ quantity that is generally negligible. The remainder of the sum may be similarly ignored as higher order time derivatives of L_α are successively smaller by factors of β^2, leading to further simplification of Eq. (36) to

$$e^{-\int^t (D-i\lambda)dt'}\int_0^t e^{\int^{t'}(D-i\lambda)dt''}\frac{\beta DK_\alpha}{D+i\lambda}dt'$$
$$\simeq \frac{\beta DK_\alpha}{D^2+\lambda^2}\bigg|_{t'=t} + \mathcal{O}(\beta^3), \tag{39}$$

and likewise for its complex conjugate in Eq. (34).

The last real term in the Eq. (34) may be shown to reduce to

$$\frac{4D}{\lambda}e^{-\int^t 2Ddt'}\int_0^t e^{\int^{t'}2Ddt''}\frac{\beta\lambda DK_\alpha}{D^2+\lambda^2}dt'$$
$$\simeq \frac{2\beta DK_\alpha}{D^2+\lambda^2}\bigg|_{t'=t} + \left[\sum_{n=1}^N \left(\frac{-1}{2D}\right)^n \frac{2\beta D}{D^2+\lambda^2}\frac{d^{(n)}}{dt'^{(n)}}K_\alpha\right]_{t'=t}$$
$$-\frac{2D}{\lambda}e^{-\int^t 2Ddt'}\int_0^t e^{\int^{t'}2Ddt''}\left(\frac{-1}{2D}\right)^N \frac{\beta\lambda}{D^2+\lambda^2}\frac{d^{(N+1)}}{dt'^{(N+1)}}K_\alpha dt'$$
$$\simeq \frac{2\beta DK_\alpha}{D^2+\lambda^2}\bigg|_{t'=t} + \mathcal{O}(\beta^3), \tag{40}$$

by similar arguments such that Eqs. (36) and (40) combine to produce

$$P_y(t) \simeq -\frac{\beta D}{D^2+\lambda^2}P_z(t) + \mathcal{O}(\beta^3). \tag{41}$$

Equation (41) clearly shows the miraculous cancellation between terms pertaining to a nonzero repopulation function originally present in Eq. (34) to first order in β. The final expression for $P_y(t)$ is therefore identical to the standard adiabatic result given by Eq. (13) in the limit of interest.

Our next task is to verify that the assumption of instantaneous repopulation in Eq. (15) is indeed correct. For this purpose, we begin by expressing P_z as a function of P_0 by way of Eq. (31),

$$P_z(t) = -\left(1+\frac{\Lambda_3}{D}\right)P_0(t) + \sum_{j=1,2,4}\frac{\beta(D+\Lambda_j)(\Lambda_3-\Lambda_j)}{\lambda\Lambda_j(2D+\Lambda_j)}e^{\int^t \Lambda_j(t')dt'}\mathcal{I}_i. \tag{42}$$

To the lowest order in β, Eq. (42) is equivalently

$$P_z(t) = -P_0(t) + \frac{4(D^2+\lambda^2)}{\beta\lambda}e^{-\int^t 2Ddt'}\int_0^t e^{\int^{t'}2Ddt''}\frac{\beta\lambda DK_\alpha}{D^2+\lambda^2}dt' + \mathcal{O}(\beta^2), \tag{43}$$

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on which we apply the same technique of integration by parts in conjunction with the rationale employed previously to eliminate the various time derivatives to obtain

$$P_z(t) \simeq -P_0(t) + 2K_\alpha(t) + 2\left[\sum_{n=1}^{N}\left(\frac{-1}{2D}\right)^n \frac{d^{(n)}}{dt'^{(n)}}K_\alpha\right]_{t'=t}$$
$$-\frac{2(D^2+\lambda^2)}{\beta\lambda}e^{-\int^t 2Ddt'}\int_0^t e^{\int^{t'} 2Ddt''}\left(\frac{-1}{2D}\right)^N \frac{\beta\lambda}{D^2+\lambda^2}\frac{d^{(N+1)}}{dt'^{(N+1)}}K_\alpha dt'$$
$$\simeq -P_0(t) + 2K_\alpha(t) + \mathcal{O}(\beta^2). \tag{44}$$

The last approximate equality in Eq. (44) is identically

$$\frac{N_\alpha(p)}{N^{\mathrm{eq}}(p,0)} \simeq K_\alpha \equiv \frac{N^{\mathrm{eq}}(p,\mu)}{N^{\mathrm{eq}}(p,0)}, \tag{45}$$

which clearly affirms the validity of the instantaneous repopulation assumption to the lowest order in β.

V. CONCLUSION

We have taken the exact four-component QKEs for a two-flavour active–sterile neutrino system and derived from them an approximate evolution equation for the relic neutrino asymmetry in which the role of repopulation is methodically embedded. The consequence of including a finite refilling function is to generate higher order terms which are readily discarded in the high temperature epoch of interest, thereby yielding a rate equation identical to that found earlier in the standard adiabatic limit where the said function is taken to be negligible. The formal procedure developed in the present work to establish this result has been labelled the extended adiabatic approximation.

Frequently adopted in numerical studies with notable accuracy, the assumption of instantaneous repopulation is also shown to arise naturally in the extended adiabatic limit. We have thus furnished a rigorous justification for the superficially incompatible assumptions of a negligible refilling function and instantaneous repopulation in the regime where collisions are the dominant asymmetry generation mode.

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